

Consistent estimators of the smoothing parameter in the Hodrick-Prescott filter

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Abstract

The so-called Hodrick-Prescott filter was first introduced in actuarial science to estimate trends from claims data and now is widely used in economics and finance to estimate and predict e.g. business cycles and trends in financial data series. This filter depends on a smoothing parameter. We propose new consistent estimators of this smoothing parameter and construct corresponding non-asymptotic confidence intervals with a precise confidence level.

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1 Introduction

Leser (1961) and later on Hodrick and Prescott (1980) and (1997) defined a trend $y = (y_1, \dots, y_T)$ of a time series $x = (x_1, \dots, x_T)$ as the minimizer of $\sum_{t=1}^T (x_t - y_t)^2 + \alpha \sum_{t=1}^{T-2} (y_{t+2} - 2y_{t+1} + y_t)^2$ for an appropriately chosen positive parameter α , called the smoothing parameter. Hodrick and Prescott (1997) suggested that when the time series x represents the log of U.S. real GNP, $\alpha = 1600$, is ought to be optimal to get a smooth trend close to the actual real GNP. This procedure for filtering a trend as a smooth curve from the data has a long history in actuarial

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science starting with the graduation method developed by Whittaker (1923) and Henderson (1924) that was the starting point for the actual filter that was first proposed by Leser (1961) and later on was introduced into economics by Hodrick and Prescott (1980). See e.g. Stigler (1978) and Green and Silverman (1994) for a nice overview.

There are two different approaches (see e.g. Green and Silverman (1994)) to the question of choosing the smoothing parameter. The first approach is to regard the free choice of smoothing parameter as an advantage. The features of the data can be explored by varying the smoothing parameter α . The other view advocates for a need of an automatic estimation method. Cross-validation and generalized cross-validation techniques are among the automatic methods used to estimate the smoothing parameter α (see Craven and Wahba (1978) and Green and Silverman (1994) for further details). After carefully documenting the shortcomings of recursive estimation methods like the Kalman filter, Schlicht (2006) proposed another automatic method, where the estimator of α is a fixed point of a highly nonlinear equation, obtained using the maximum-likelihood and moments matching techniques. The consistency of Schlicht's estimator is however still an open problem.

Our contribution should be seen as a continuation of the work by Schlicht (2006) in trying to find a systematic way to estimate the smoothing parameter α with *consistent* statistics, still for the technically most tractable Gaussian random walk model of the trend. The main result of the paper suggests a new family of estimators of the smoothing parameter α which are explicit and consistent. We also construct a confidence interval with a precise confidence level.

In Section 2, we introduce the Hodrick-Prescott filter and in Section 3, we derive consistent estimators of the smoothing parameter α , when the trend is a Gaussian random walk. Finally, in Section 4, we apply our estimators to extract the trend associated with a set of market data including the US real GNP, the spot prices of the major currencies and equity indices. The quite good fit of the trend to the considered time series, suggests that this approach to trend estimation may be useful for practitioners, due to the relatively simple form of the estimators.

2 The Hodrick-Prescott filter

Let $x = (x_1, \dots, x_T) \in \mathbb{R}^T$ be a time series of observables. The Hodrick-Prescott filter (HP in short) decomposes x into a nonstationary trend $y \in \mathbb{R}^T$ and a cyclical residual component $u \in \mathbb{R}^T$:

$$x = y + u. \tag{2.1}$$

Given a smoothing parameter $\alpha > 0$, this decomposition of x is obtained by minimizing the weighted sum of squares

$$\|x - y\|^2 + \alpha \|D^2 y\|^2 \tag{2.2}$$

with respect to y , where for $a \in \mathbb{R}^T$, $\|a\|^2 = \sum_{i=1}^T a_i^2$.

Here, D^2y is the trend disturbance obtained by acting the second order forward shift operator D^2 on the trend $y = (y_1, y_2, \dots, y_T)$:

$$D^2y_t := (y_{t+2} - y_{t+1}) - (y_{t+1} - y_t), \quad t = 1, 2, \dots, T - 2,$$

or, equivalently,

$$D^2y_t := 2 \left(\frac{y_{t+2} + y_t}{2} - y_{t+1} \right), \quad t = 1, 2, \dots, T - 2,$$

measuring the deviation between the value of the trend at $t + 1$, y_{t+1} and the linear interpolation between y_t and y_{t+2} .

In vector form,

$$Py(t) = D^2y_t, \quad t = 1, \dots, T - 2, \quad (2.3)$$

where, the shift operator P is the following $(T - 2) \times T$ -matrix

$$P := \begin{pmatrix} 1 & -2 & 1 & \dots & \dots & 0 \\ 0 & 1 & -2 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 1 & -2 & 1 \end{pmatrix}.$$

As pointed out e.g. in Pedersen (2001), the first term in (2.2) measures a goodness-of-fit by minimizing the the deviation between the trend y_t and the observation x_t and the second term is a measure of the degree-of-smoothness which penalizes decelerations in growth rate of the trend component, by minimizing the the deviation between the trend value y_{t+1} and the linear interpolation between y_t and y_{t+2} .

Both Reeves *et al.* (2000) and Araujo *et al.* (2003), found the third order shift operator more appropriate than the second order one to extract the trend from some Foreign Exchange Rate series. The matrix formulation to the HP-filter adopted in this paper fully extends to higher order shift operators or any other smoothing operator.

Since P is of rank $T - 2$, $Py = v$ does not determine a unique y but rather the set of solutions (see Schlicht (2006) for further details)

$$y := \{P'(PP')^{-1}v + Z\beta; \beta \in \mathbb{R}^2\}$$

where the $T \times 2$ -matrix Z satisfies

$$PZ = 0, \quad Z'Z = I_2, \quad (2.4)$$

with I_2 denoting the 2×2 identity matrix. In view of Eq. (2.1), the time series x can be represented in terms of (u, v) as

$$x = u + P'(PP')^{-1}v + Z\beta, \quad (2.5)$$

for some $\beta \in \mathbb{R}^2$.

As the matrix $(I_T + \alpha P'P)$ is positive definite, the unique solution $y(\alpha, x)$ to the optimal problem (2.2) is

$$y(\alpha, x) = (I_T + \alpha P'P)^{-1}x, \quad (2.6)$$

where I_T denotes the $T \times T$ identity matrix. Eq. (2.6) defines the descriptive filter that associates a trend y to the time series x , depending on the smoothing parameter α and the disturbance operator P .

Following Schlicht (2006), a way to estimate the smoothing parameter α is to let the optimal solution $y(\alpha, x)$ in Eq. (2.6) be the best predictor of any trend y given the time series x , i.e.

$$y(\alpha, x) = E[y|x]. \quad (2.7)$$

This approach of estimating α assumes that we are able to compute explicitly this conditional expectation, which is not always the case. The normal and more generally the elliptical probability distributions are among the few models for which an explicit formula for the conditional expectation is possible. In order to estimate the trend and the smoothing parameter, given the time series of observations x , we obviously need a model for the joint distribution of (x, y) . Using (2.3) and (2.5), this can be achieved through imposing a model for the joint distribution of (u, v) .

3 A Gaussian random walk model of the trend

In the literature (cf. e.g. Hodrick and Prescott (1997) and Schlicht (2006)), a widely used model (and perhaps the only feasible case) for the joint distribution of (u, v) , is to assume that the disturbances u and v independent and normally distributed. This turns (x, y) into a normally distributed vector, which makes the estimation issue of the trend y and the smoothing parameter α , using (2.6) and (2.7), feasible.

3.1 Independent and identically distributed disturbances

In particular, assuming furthermore that u and v have zero means and covariance matrices $\sigma_u^2 I_T$ and $\sigma_v^2 I_{T-2}$, where I_T and I_{T-2} denote the $T \times T$ and $(T-2) \times (T-2)$ identity matrices, respectively:

$$\begin{pmatrix} u \\ v \end{pmatrix} \sim \mathcal{N}(0, \Sigma_{uv}), \quad (3.1)$$

with covariance matrix

$$\Sigma_{uv} := \begin{pmatrix} \sigma_u^2 I_T & 0 \\ 0 & \sigma_v^2 I_{T-2} \end{pmatrix},$$

makes the increments of the trend y following a Gaussian random walk, since, by Eq. (2.3), $y_{t+2} - y_{t+1} = y_{t+1} - y_t + v_t$. This turns the time series x into a trend

y generated by a Gaussian random walk and a normal disturbance u . That is, in view of (2.5), (x, y) is normally distributed:

$$\begin{pmatrix} x \\ y \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} Z \\ Z \end{pmatrix} \beta, \Sigma_{xy}\right), \quad (3.2)$$

with covariance matrix

$$\Sigma_{xy} := \begin{pmatrix} \sigma_u^2 I_T + \sigma_v^2 Q & \sigma_v^2 Q \\ \sigma_v^2 Q & \sigma_v^2 Q \end{pmatrix},$$

where,

$$Q := P'(PP')^{-1}(PP')^{-1}P.$$

This yields an explicit expression of the conditional expectation of the trend y given the time series x :

$$E[y|x] = Z\beta + \sigma_v^2 Q [\sigma_u^2 I_T + \sigma_v^2 Q]^{-1} (x - Z\beta). \quad (3.3)$$

Now, in view of (2.7), the smoothing parameter α and the parameter β satisfy

$$Z\beta + \sigma_v^2 Q [\sigma_u^2 I_T + \sigma_v^2 Q]^{-1} (x - Z\beta) = (I_T + \alpha P'P)^{-1} x. \quad (3.4)$$

In the following proposition, we give a characterization of the optimal smoothing parameter α and the free parameter β that satisfy Eq. (3.4) in terms of the time series of observations x , the second order forward shift operator P and Z . This is a slight generalization of Theorem 1 in Schlicht (2006).

Proposition 3.1

- (a) If Equation (3.4) holds then $\beta = Z'x$.
- (b) Equation (3.4) holds if and only if $\beta = Z'x$ and $\alpha = \sigma_u^2/\sigma_v^2$.

Proof.

(a) Multiplying both terms in Eq. (3.4) by $(I_T + \alpha P'P)$ and ordering terms, using $PZ = 0$, gives

$$(I_T + \alpha P'P)\sigma_v^2 Q [\sigma_u^2 I_T + \sigma_v^2 Q]^{-1} (x - Z\beta) = x - Z\beta.$$

Noting that $QZ = Z'Q = 0$, $Z'Z = I_2$ and that

$$P'PQ = I_T - ZZ', \quad (3.5)$$

we get

$$\left(\left(\alpha - \frac{\sigma_u^2}{\sigma_v^2}\right)I_T - \alpha ZZ'\right) \left[\frac{\sigma_u^2}{\sigma_v^2}I_T + Q\right]^{-1} (x - Z\beta) = 0.$$

Now, using

$$ZZ' \left[\frac{\sigma_u^2}{\sigma_v^2} I_T + Q \right]^{-1} = \frac{\sigma_u^2}{\sigma_v^2} ZZ', \quad (3.6)$$

because $Z'Q = 0$, and rearranging terms, we get

$$\left(\alpha - \frac{\sigma_u^2}{\sigma_v^2} \right) \left[\frac{\sigma_u^2}{\sigma_v^2} I_T + Q \right]^{-1} (x - Z\beta) = \alpha \frac{\sigma_u^2}{\sigma_v^2} ZZ' (x - Z\beta). \quad (3.7)$$

Multiplying both terms by ZZ' and using Eq. (3.6) and $Z'Z = I_2$ yields

$$\frac{\sigma_u^2}{\sigma_v^2} ZZ' (x - Z\beta) = 0,$$

and then

$$\beta = Z'x.$$

(b) Setting $\beta = Z'x$ into Eq. (3.7) and using (3.5) yields

$$\left(\alpha - \frac{\sigma_u^2}{\sigma_v^2} \right) P'PQx = 0.$$

It remains to show that $P'PQx \neq 0$.

Indeed, since

$$P'PQx = P'(PP')^{-1}Px,$$

and the matrix $P'(PP')^{-1}P \neq 0$ then the Gaussian vector $P'PQx$ does not vanish almost surely, which yields

$$\alpha = \frac{\sigma_u^2}{\sigma_v^2}.$$

□

Remark 3.2 The estimator $\hat{\beta} = Z'x$ of the parameter β is derived in Schlicht (2006) as the maximum likelihood estimator based on the likelihood function of the time series x .

3.2 Consistent estimators of the variances σ_u^2 and σ_v^2

In this section we propose explicit unbiased consistent estimators of the variances σ_u^2 and σ_v^2 , in the sense that, as the length T of the series of observations tends to infinity, the proposed estimators converge in probability, with normally distributed errors to the respective variances. A consistent estimator of the smoothing parameter α will be the ratio of the estimators of the respective variances of u and v .

To this end, we use observations from the centered time series Px :

$$Px = v + Pu \sim \mathcal{N}(0, \sigma_v^2 I_{T-2} + \sigma_u^2 PP')$$

First we note that the Gaussian time series Px is stationary. In fact the elements $V(i, j)$ of the $(T - 2 \times T - 2)$ covariance matrix of Px satisfy

$$V(i, j) = \sigma_v^2 \delta_i^j + \sigma_u^2 (PP')_{ij} = r_{|i-j|},$$

where,

$$r_k = \begin{cases} \sigma_v^2 + 6\sigma_u^2, & \text{if } k = 0; \\ -4\sigma_u^2, & \text{if } k = 1; \\ \sigma_u^2, & \text{if } k = 2; \\ 0, & \text{otherwise.} \end{cases}$$

Now, it is a classical fact that (see e.g. Chapter 7 in Brockwell and Davis (1991) or Proposition 2.1 in Giurcanu and Spokoiny (2002))

$$\hat{r}_k = \frac{1}{(T-2)-k} \sum_{j=1}^{T-2-k} Px(j)Px(j+k), \quad k = 0, 1, 2,$$

is an unbiased estimator of $r_k = E[Px(s)Px(s+k)]$ in the sense that $E[\hat{r}_k] = r_k$.

We want an explicit form of the quadratic error

$$a_k^2 := \text{Var}(\hat{r}_k) \quad k = 0, 1, 2.$$

To this purpose, let A_k be the $(T-2) \times (T-2)$ matrix with the entries

$$\frac{1}{2(T-2-k)} (\delta(s-t-k) + \delta(t-s-k)).$$

We recall the following result due to Giurcanu and Spokoiny (2002).

Proposition 3.3 *We have, for $k = 0, 1, 2$,*

$$\hat{r}_k = (Px)^t A_k Px, \quad r_k = \text{tr}(A_k V)$$

and

$$a_k^2 = 2\text{tr}(A_k V)^2.$$

By a simple computation we have for $T > 4$,

$$a_0^2 = \frac{1}{(T-2)^2} [2(T-2)r_0^2 + 4(T-3)r_1^2 + 4(T-4)r_2^2], \quad (3.8)$$

$$a_1^2 = \frac{1}{(T-3)^2} [(T-3)(r_1^2 + r_0^2) + 2(T-4)(r_2 r_0 + r_1^2) + 2(T-5)r_2^2] \quad (3.9)$$

and

$$a_2^2 = \frac{1}{(T-4)^2} [(T-4)(r_2^2 + r_0^2) + 2(T-5)r_1^2 + 2(T-6)r_2^2]. \quad (3.10)$$

From that we derive the following explicit central limit for the estimators \hat{r}_k , $k = 0, 1, 2$.

Corollary 3.4

- (1)
$$\sqrt{T}[\hat{r}_0 - r_0] \rightarrow N(0, 2r_0^2 + 4r_1^2 + 4r_2^2).$$
- (2)
$$\sqrt{T}[\hat{r}_1 - r_1] \rightarrow N(0, [(r_0 + r_2)^2 + r_2^2 + 3r_1^2]).$$
- (3)
$$\sqrt{T}[\hat{r}_2 - r_2] \rightarrow N(0, r_0^2 + 2r_1^2 + 3r_2^2).$$

The easily checked relation

$$E[\hat{r}_1] = -4\sigma_u^2$$

suggests the following consistent unbiased estimator of σ_u^2

$$\hat{\sigma}_u^2 = -\frac{1}{4}\hat{r}_1 = -\frac{1}{4(T-3)} \sum_{j=1}^{T-3} Px(j)Px(j+1). \quad (3.11)$$

The relation

$$E[\hat{r}_2] = \sigma_u^2$$

suggests also another estimator of σ_u^2 :

$$\tilde{\sigma}_u^2 = \hat{r}_2 = \frac{1}{(T-4)} \sum_{j=1}^{T-4} Px(j)Px(j+2). \quad (3.12)$$

Furthermore, since

$$E[\hat{r}_0] = \sigma_v^2 + 6\sigma_u^2,$$

in view of (3.11) and (3.12), we get two consistent unbiased estimators of σ_v^2 :

$$\hat{\sigma}_v^2 = \hat{r}_0 + \frac{3}{2}\hat{r}_1,$$

or

$$\hat{\sigma}_v^2 = \frac{1}{T-2} \sum_{j=1}^{T-2} Px(j)^2 + \frac{3}{2(T-3)} \sum_{j=1}^{T-3} Px(j)Px(j+1), \quad (3.13)$$

and

$$\tilde{\sigma}_v^2 = \hat{r}_0 - 6\hat{r}_2,$$

that is,

$$\tilde{\sigma}_v^2 = \frac{1}{T-2} \sum_{j=1}^{T-2} Px(j)^2 - \frac{6}{(T-4)} \sum_{j=1}^{T-4} Px(j)Px(j+2), \quad (3.14)$$

Therefore, in view of Proposition 3.1, we get two non-negative consistent estimators of the positive smoothing parameter α :

$$\hat{\alpha} = \max \left\{ 0, -\frac{1}{4} \left(\frac{3}{2} + \frac{(T-3) \sum_{j=1}^{T-2} Px(j)^2}{(T-2) \sum_{j=1}^{T-3} Px(j)Px(j+1)} \right)^{-1} \right\}$$

and

$$\tilde{\alpha} = \max \left\{ 0, \left(\frac{(T-4) \sum_{j=1}^{T-2} Px(j)^2}{(T-2) \sum_{j=1}^{T-4} Px(j)Px(j+2)} - 6 \right)^{-1} \right\}.$$

Proposition 3.5 *The following non-negative statistics*

$$\hat{\alpha} = \max \left\{ 0, -\frac{1}{4} \left(\frac{3}{2} + \frac{(T-3) \sum_{j=1}^{T-2} Px(j)^2}{(T-2) \sum_{j=1}^{T-3} Px(j)Px(j+1)} \right)^{-1} \right\} \quad (3.15)$$

and

$$\tilde{\alpha} = \max \left\{ 0, \left(\frac{(T-4) \sum_{j=1}^{T-2} Px(j)^2}{(T-2) \sum_{j=1}^{T-4} Px(j)Px(j+2)} - 6 \right)^{-1} \right\}, \quad (3.16)$$

based on the time series of observation Px , are consistent estimators of the smoothing parameter α .

3.3 Non-asymptotic confidence intervals for r_0 and r_1

We will only treat the cases r_0, r_1 because $r_1 = -4r_2$. We have the following

Proposition 3.6 *We have for $k = 0, 1$, $T \geq 4$, and $\lambda \leq \frac{2a_k}{3\|A_k V\|_\infty}$,*

$$P(|\hat{r}_k - r_k| \geq a_k \lambda) \leq \exp\left(-\frac{\lambda^2}{4}\right).$$

Proof. We use the following general result for Gaussian quadratic forms (see e.g. Giurcanu and Spokoiny (2002)). Let Y be a centered Gaussian vector with covariance matrix C and $a^2 = \text{var}(Y^t A Y)$. Then it holds that

$$P(|Y^t A Y - \mathbf{E}(Y^t A Y)| \geq \lambda a) \leq \max\left\{\exp\left(-\frac{\lambda^2}{4}\right), \exp\left(-\frac{\lambda a}{6\|C A\|_\infty}\right)\right\}.$$

Therefore, from the representation $\hat{r}_k = (Px)^t A_k Px$ we get that

$$P(|(Px)^t A_k Px - \mathbf{E}((Px)^t A_k Px)| \geq \lambda a_k) \leq \max\left\{\exp\left(-\frac{\lambda^2}{4}\right), \exp\left(-\frac{\lambda a_k}{6\|V A_k\|_\infty}\right)\right\}.$$

which achieves the proof. \square

Now we estimate precisely the quantity $\|VA_k\|_\infty$. Indeed, we have

$$A_0V = \frac{1}{(T-2)}V$$

Hence,

$$\|A_0V\|_\infty = \frac{r_0}{T-2}.$$

Now,

$$A_1V(i, j) = \frac{1}{2(T-3)}[V(i-1, j) + V(i+1, j)]$$

implies that

$$\|A_1V\|_\infty = (T-3) \max(r_0 + r_2, 2|r_1|).$$

Proposition 3.7

(1) If $T > 2$ and $\lambda \leq \frac{\sqrt{8(T-2)}}{3}$, then

$$P(|\hat{r}_0 - r_0| \geq a_0\lambda) \leq \exp\left(-\frac{\lambda^2}{4}\right).$$

(2) If $T > 3$ and $\lambda \leq \frac{\sqrt{(T-3)}}{3}$, then

$$P(|\hat{r}_1 - r_1| \geq a_1\lambda) \leq \exp\left(-\frac{\lambda^2}{4}\right).$$

(3) If $T > 3$ and $\lambda \leq \frac{\sqrt{(T-3)}}{3}$, then

$$P\left(\max_{k=0,1} |\hat{r}_k - r_k| \geq \frac{r_0\lambda\sqrt{10}}{\sqrt{T-3}}\right) \leq 2 \exp\left(-\frac{\lambda^2}{4}\right).$$

The third estimate follows from the fact that $\max(a_0^2, a_1^2) \leq \frac{10r_0^2}{T-3}$.

Now we can give the following confidence intervals for r_0 .

Proposition 3.8 Let $T > 2$ and $\lambda \leq \frac{\sqrt{8(T-2)}}{3}$ then

$$r_0 \in \left[\frac{\hat{r}_0}{1 + \frac{\lambda\sqrt{10}}{\sqrt{T-2}}}, \frac{\hat{r}_0}{1 - \frac{\lambda\sqrt{10}}{\sqrt{T-2}}} \right]$$

with probability at least $1 - \exp\left(-\frac{\lambda^2}{4}\right)$.

Proof. Using Estimate (1) in Proposition 3.7, the confidence interval follows from the fact that $a_0^2 \leq \frac{10r_0^2}{(T-2)}$. \square

Now we suppose we are going to give a confident interval for r_1 which depends on α . First, since $\sigma_u^2 = \alpha\sigma_v^2$ we have $r_0 = -\frac{(1+6\alpha)}{4}r_1$ implying

$$a_1^2 \leq \frac{[\frac{(1+6\alpha)^2}{16} + 1 + 2(\frac{(1+6\alpha)}{16} + 1)r_1^2 + 2r_1^2\frac{(1+6\alpha)^2}{16}]}{T-3}$$

or

$$a_1^2 \leq \frac{(3(1+6\alpha)^2 + 2(1+6\alpha) + 48)r_1^2}{16(T-3)}.$$

From this it follows that

$$a_1 \leq -\sqrt{\frac{(3(1+6\alpha)^2 + 2(1+6\alpha) + 48)}{16(T-3)}}r_1.$$

Now we can announce our confident interval for r_1 .

Proposition 3.9 *Let $T > 3$, and $\lambda \leq \frac{\sqrt{(T-3)}}{3}$ then*

$$r_1 \in \left[\frac{\hat{r}_1}{1 - \frac{\lambda\sqrt{3(1+6\alpha)^2 + 2((1+6\alpha)+48)}}{4\sqrt{T-3}}}, \frac{\hat{r}_1}{1 + \frac{\lambda\sqrt{3(\alpha+6)^2 + 2((1+6\alpha)+48)}}{4\sqrt{T-3}}} \right].$$

with probability at least $1 - \exp(-\frac{\lambda^2}{4})$.

Remark 3.10 Using $r_0 = \sigma_v^2(1+6\alpha)$ and $\sigma_u^2 = \alpha\sigma_v^2$ we have the following confident intervals. If $T > 3$ and $\lambda \leq \frac{\sqrt{(T-3)}}{3}$ then

$$\sigma_v^2(1+6\alpha) \in \left[\frac{\hat{r}_0}{1 + \frac{\lambda\sqrt{10}}{\sqrt{T-2}}}, \frac{\hat{r}_0}{1 - \frac{\lambda\sqrt{10}}{\sqrt{T-2}}} \right]$$

and

$$\alpha\sigma_v^2 \in \left[-\frac{\hat{r}_1}{4 + \frac{\lambda\sqrt{3(1+6\alpha)^2 + 2((1+6\alpha)+48)}}{\sqrt{T-3}}}, -\frac{\hat{r}_1}{4 - \frac{\lambda\sqrt{3(\alpha+6)^2 + 2((1+6\alpha)+48)}}{\sqrt{T-3}}} \right],$$

with probability at least $1 - 2\exp(-\frac{\lambda^2}{4})$.

If $\lambda = \frac{1}{3}\sqrt{\ln(T-3)}$, then $\exp(-\frac{\lambda^2}{4}) = \frac{1}{(T-3)^{36}}$, and the latter probability is equal to $1 - \frac{2}{(T-3)^{36}}$.

4 Numerics and calibration

In this section we give a numerical illustration of the performance of the estimators $\hat{\alpha}$ and $\tilde{\alpha}$, and construct a trend from market data including the quarterly US real GNP monthly spot prices of the major currencies and equity indexes. Needless to mention that although both estimators are consistent, which is the main novelty in this paper, their dispersion may be large, in which case the corresponding statistic is not good enough for estimating the true value of the positive smoothing parameter α . In fact, as it is easily checked, large simulations drawn from the normal distribution of $Px \sim \mathcal{N}(0, \sigma_v^2 I_{T-2} + \sigma_u^2 PP')$ and specifying values of σ_u^2 and σ_v^2 , show that, unlike $\hat{\alpha}$, the values of $\tilde{\alpha}$ being zero arise quite often, which may indicate that this statistic may not be a good estimator of the true value of α . We will therefore construct a trend y from the above mentioned market data based on $\hat{\alpha}$ only.

The market data used for this exercise include:

- The quarterly US real GNP (at a log scale) under the period January 1, 1947 – January 1, 2006;
- The monthly British Pound spot price (GBP) under the period January 29, 1971 – July 31, 2006;
- The monthly Euro spot price (Euro) under the period December 31, 1998 – July 31, 2006;
- The monthly Swiss Franc spot price (CHF) under the period January 29, 1971 – July 31, 2006;
- The monthly Japanese Yen spot price (JPY) under the period January 29, 1971 – July 31, 2006;
- The monthly S&P 500 spot price (S&P) under the period August 31, 1956 – July 31, 2006;
- The monthly Nasdaq 100 spot price (Nasdaq) under the period February 28, 1985 – July 31, 2006;
- The monthly Dow Jones Euro stoxx 500 spot price (DJ) under the period December 31, 1985 – July 31, 2006;
- The monthly FTSE 100 spot price (FTSE) under the period January 31, 1984 – July 31, 2006.
- The monthly NIKKEI 225 spot price (Nikkei) under the period January 30, 1984 – July 31, 2006.

In Tables 1 and 2, we collect the estimated values of the statistics $\hat{\alpha}$ and the largest and the smallest relative value of the cyclical component $u = x - y$, for each of the time series:

$$c^* = \max_t \left(\frac{x(t) - y(t)}{x(t)} \right), \quad c_* = \min_t \left(\frac{x(t) - y(t)}{x(t)} \right).$$

x	US real GNP	GBP	Euro	CHF	JPY (log scale)
$\hat{\alpha}$	1.05	152.25	2.13	315.32	71.36
c^*	0.11%	12%	3.4%	11%	1.6%
c_*	-0.2%	-14%	-3.3%	-14%	-2%

Table 1: Estimated values of the smoothing parameter α .

x	S&P 500	Nasdaq	Dow Jones	FTSI	Nikkei
$\hat{\alpha}$	215.20	131.03	51.53	52.68	207.58
c^*	2.6%	3%	2.1%	2%	1.6%
c_*	-4.8%	-4.6%	-2.1%	-2.4%	-2.5%

Table 2: Estimated values of the smoothing parameter α .

As shown in the figures below (see also the exhibits in the appendix), we get a quite good fit of the trend to the considered time series, which suggests that this approach to trend estimation may be useful for practitioners. Unlike Reeves *et al.* (2000) and Araujo *et al.* (2003), that found the third order shift operator more appropriate than the second order one to extract the trend from some Foreign Exchange Rate series, our estimator of the trend shows a good fit, using only the second order shift operator.

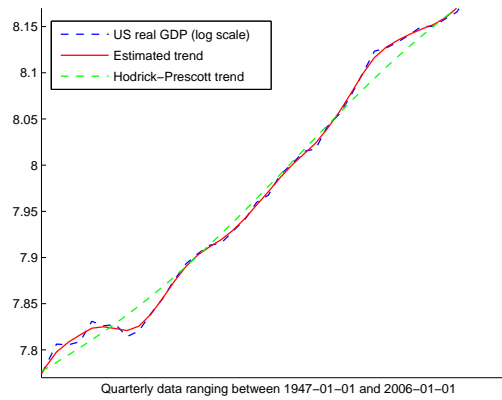


Figure 1: The estimated trend fitted with the quarterly US real GNP under the period 01-01-1947 to 01-01-2006. The Hodrick-PreScott trend corresponds to $\alpha = 1600$.

Source: U.S. Department of Commerce: Bureau of Economic Analysis.

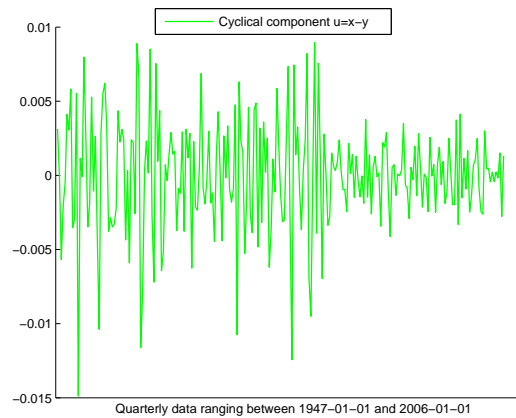


Figure 2: The cyclical component of the quarterly US real GNP under the period between 01-01-1947 and 01-01-2006.

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Appendix

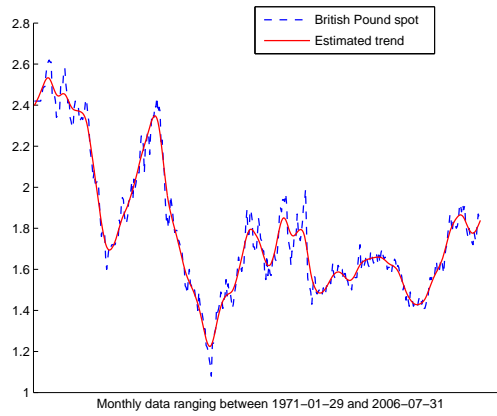


Figure 3: The estimated trend fitted with the monthly British Pound spot price under the period January 29, 1971 – July 31, 2006.

Source: Bloomberg.

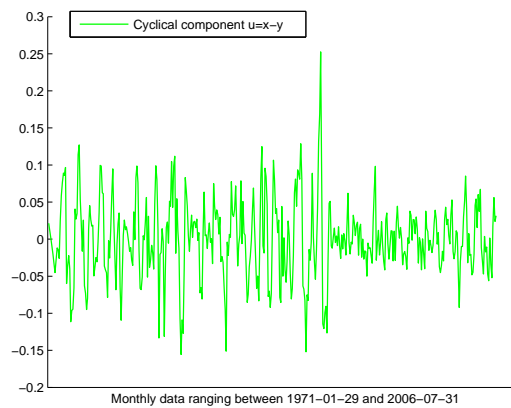


Figure 4: The cyclical component of the monthly British Pound spot price under the period January 29, 1971 – July 31, 2006.

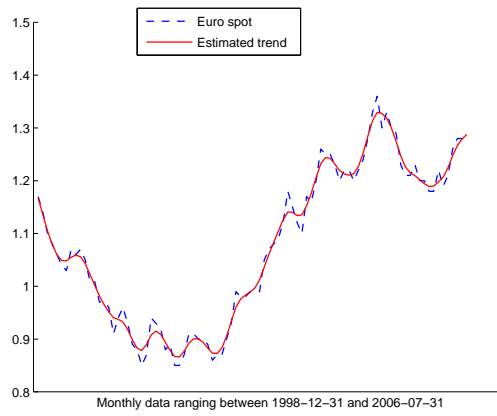


Figure 5: The estimated trend fitted with the monthly Euro spot price under the period December 29, 1998 – July 31, 2006.

Source: Bloomberg.

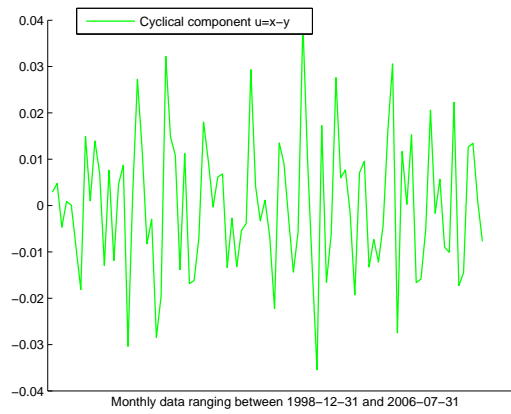


Figure 6: The cyclical component of the monthly Euro spot price under the period December 29, 1998 – July 31, 2006.

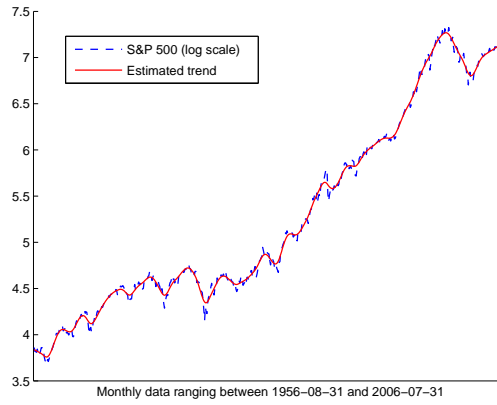


Figure 7: The estimated trend fitted with the monthly S&P 500 spot price under the period August 31, 1956 – July 31, 2006.

Source: Bloomberg.

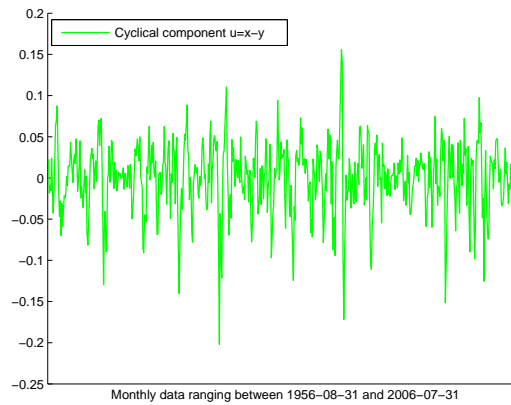


Figure 8: The cyclical component of the monthly S&P 500 spot price under the period August 31, 1956 – July 31, 2006.

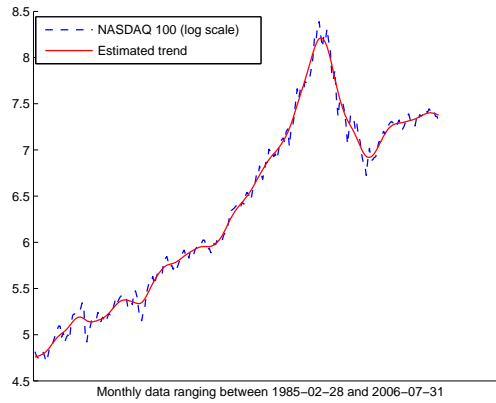


Figure 9: The estimated trend fitted with the monthly NASDAQ 100 price under the period February 28, 1985 – July 31, 2006.

Source: Bloomberg.

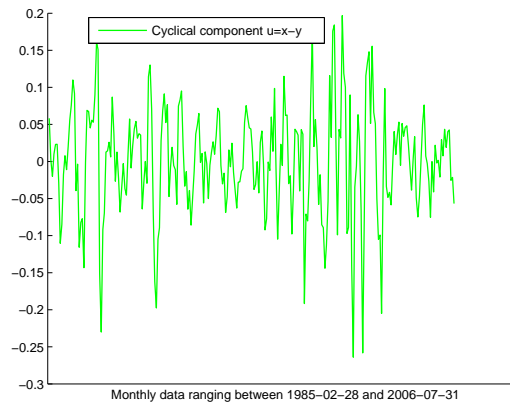


Figure 10: The cyclical component of the monthly NASDAQ 100 spot price under the period February 28, 1985 – July 31, 2006.

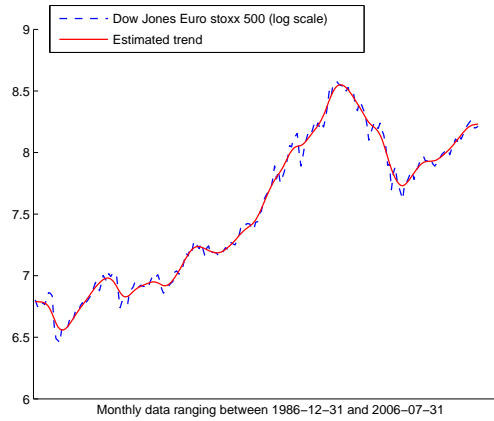


Figure 11: The estimated trend fitted with the monthly Dow Jones Euro Stoxx 500 spot price under the period December 31, 1986 – July 31, 2006.

Source: Bloomberg.

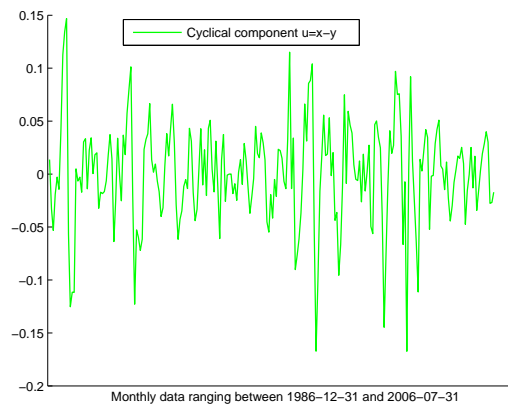


Figure 12: The cyclical component of the monthly Dow Jones Euro Stoxx 500 spot price under the period December 31, 1986 – July 31, 2006.

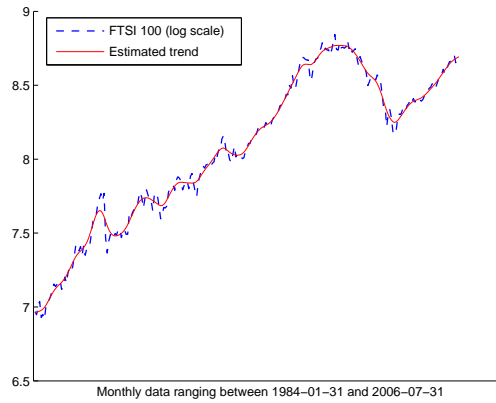


Figure 13: The estimated trend fitted with the monthly FTSE 100 spot price under the period January 31, 1984 – July 31, 2006.

Source: Bloomberg.

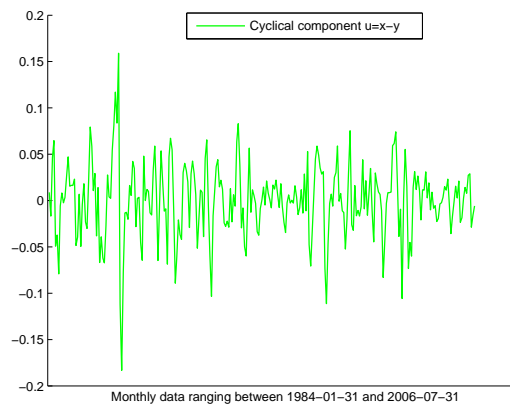


Figure 14: The cyclical component of the monthly FTSE 100 spot price under the period January 31, 1984 – July 31, 2006.

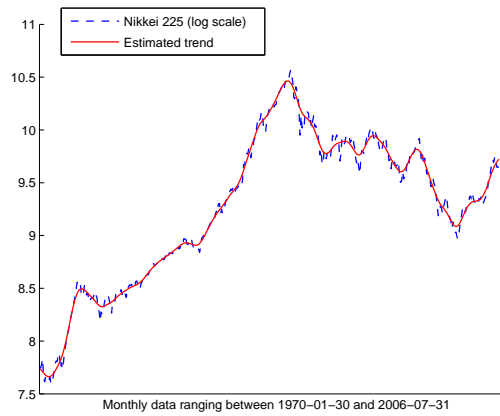


Figure 15: The estimated trend fitted with the monthly Nikkei 225 spot price under the period January 30, 1970 – July 31, 2006.

Source: Bloomberg.

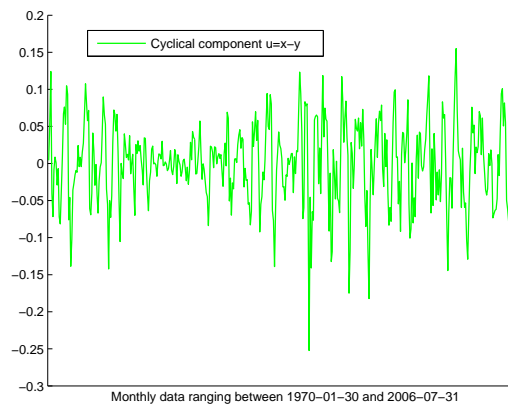


Figure 16: The cyclical component of the monthly Nikkei 225 spot price under the period January 30, 1970 – July 31, 2006.